Analytical Insights into the Modified Fractional Bell Polynomial with Mittag-Leffler Parameter

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ABSTRACT

In this paper, we will define the Modified Fractional Bell Polynomial by incorporating the Mittag Leffler function of one parameter. The Existence and convergence of the Modified Fractional Bell Polynomial will be established by extending the classical results of Bell Polynomial and Mittag Leffler function of one parameter in the fractional calculus. Additionally, we explore the inverse of the Modified Fractional Bell Polynomial, providing a step-by-step proof of its existence. This result enhances the applicability of the polynomial by allowing a unique mapping from each output to a set of input values.

The introduction of the Modified Fractional Bell Polynomial, with its well-established properties, opens avenues for further research and applications in diverse mathematical contexts. The generality of the polynomial makes it a powerful tool for modeling complex phenomena.

Keywords: Fractional Bell Polynomials; Mittag-Leffler Function; Generalization; Modified Fractional Bell Polynomial; Existence; Continuity; Convergence; Inverse Function; Mathematical Analysis; Special Functions

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INTRODUCTION

Fractional calculus is a branch of mathematical analysis that specifies the concept of differentiation and integration to non-integer orders. Traditional calculus allocates with integer-order derivatives and integrals, but fractional calculus covers these operations to include non-integer or fractional orders. This field has start applications in various scientific and engineering disciplines, such as physics, biology, control theory, and signal processing.Fractional differential equations often exhibit non-local properties, making both analytical and numerical solutions challenging. Analytical solutions may involve fractional calculus operators, and closed-form solutions may not always be readily available. Numerical solutions may require specialized algorithms to handle non-integer derivatives and integrals accurately. The detail study about the fractional development of differential and integral calculus can be found in [6, 7, 8, 9] and since the finding the analytically solution for the fractional order is not that simple so numerical approach to find the approximate solution is one of the alternate and simple method, several researcher work on numerical methods [10, 11, 12, 13, 14, 15]. Differential equations and fractional differential equations are mathematical tools used to model various physical and biological systems.

Special functions play a crucial role in fractional calculus, providing mathematical tools to express and manipulate solutions to fractional differential equations. These functions often emerge as solutions to integral and differential operators involving non-integer orders, which are inherent in fractional calculus. One such example is the Mittag-Leffler function, denoted as $E_\alpha(z)$, which frequently appears in the solutions of fractional differential equations. The Mittag-Leffler function generalizes the exponential function and is defined through a power series, making it a

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fundamental tool for expressing fractional derivatives and integrals. Special functions such as the Wright function, the fractionalorder Bessel functions, and the fractional-order Legendre functions are also employed to represent solutions in various contexts. These functions possess unique properties that are well-suited for characterizing the behavior of systems exhibiting fractional dynamics, allowing researchers and practitioners to analyze and model complex phenomena in fields such as physics, engineering, and biology.

Moreover, special functions facilitate the Laplace transform and Fourier transform techniques commonly used in fractional calculus. These transforms involve integrals and convolutions with special functions, enabling the translation of fractional differential equations into algebraic or simpler differential equations in the transformed domain. The use of special functions thus streamlines the solution process and aids in obtaining closed-form solutions for a wide range of fractional differential equations. In essence, special functions in fractional calculus serve as a rich and diverse toolbox, empowering researchers to navigate the intricacies of non-integer order operators and facilitating a deeper understanding of the behavior of systems with fractional dynamics.

In this paper first we will recall some basic definitions and results, in the next section we give some results as main results followed by the new define function, and finally application based examples in the consecutive section.

PRELIMINARY RESULTS

In this section we give some basic definitions , results and relations.

Definition 0.1. *(Mittag-Leffler Function) [33] The Mittag - Leffler function of one parameter is denoted by* $E_\alpha(z)$ *and defined as,*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} z^k \tag{0.1}$$

where $z, \alpha \in C$, $Re(\alpha) > 0$.

If we put $\alpha = 1$, then the above equation becomes

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z. \tag{0.2}$$

Definition 0.2. *(Riemann-Liouville Left sided and right-sided operator)*
Riemann-Liouville define the most popular fractional operator [18, 19, 20] in the form of left-sided operator

$${}^{RL}\mathrm{D}_{a+}^{\alpha}[f(x)] = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_a^x (x-\xi)^{n-\alpha-1} f(\xi) \mathrm{d}\xi, x \geq a$$

in the similar way the right-sided introduced was

$${}^{RL}\mathrm{D}_{b-}^{\alpha}[f(x)] = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_x^b (\xi-x)^{n-\alpha-1} f(\xi) \mathrm{d}\xi, x \leq b$$

Definition 0.3. *(Caputo left-sided and right sided operator)*
The next extended fractional operator [20, 21, 22, 23] is the Caputo left sided operator

$$^{C}{}_{+}[f(x)] = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\xi)^{n-\alpha-1} \frac{\mathrm{d}^n}{\mathrm{d}\xi^n}[f(\xi)]\mathrm{d}\xi, x \geq a \quad \mathrm{D}_a$$

and the right sided one is

$$^{c}\mathrm{D}_b^{\alpha}[f(x)] = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (\xi-x)^{n-\alpha-1} \frac{\mathrm{d}^n}{d\xi^n}[f(\xi)]\mathrm{d}\xi, x \leq b$$

Definition 0.4. *(Bell Polynomial)[16, 17] The Bell polynomial $B_{n,k}(x_1,x_2,...,x_k)$ is defined recursively as follows:*

$$B_{0,0} = 1,$$

$$B_{n+1,k} = x_{k+1}B_{n,k} + B_{n,k-1},$$

for $n \geq 0$ and $1 \leq k \leq n+1$, with the conventions $B_{n,0} = B_{n,n+1} = 0$ for all $n \geq 0$.

Here, n represents the total number of elements in a set, k is the number of non-empty subsets considered in the combinatorial structure, and $x_1,x_2,...,x_k$ are indeterminates. Bell polynomials have applications in partition theory, set partitions, and various combinatorial counting problems. They are instrumental in expressing and studying exponential generating functions, making them a valuable tool in combinatorial analysis. The recursive definition allows for efficient computation and manipulation of these polynomials in combinatorial applications.

Theorem 0.5. *(Existence and Uniqueness of Bell Polynomials)[41, 42, 43]*

For all non-negative integers n and k, the Bell polynomials $B_{n,k}$ exist and are uniquely determined by their recursive definition.

For all non-negative integers n and k, the Bell polynomials $B_{n,k}$ are defined recursively as follows:

$$B_{0,0} = 1,$$

$$B_{n+1,k} = x_{k+1}B_{n,k} + B_{n,k-1} \text{ for } n \geq 0 \text{ and } 1 \leq k \leq n+1, B_{n,0} = B_{n,n+1} = 0$$

$$\text{for all } n \geq 0.$$

Theorem 0.6. *(Convergence of Generating Functions)[41, 42, 43]*

The generating functions associated with Bell polynomials exhibit convergence properties within a certain radius of convergence, dependent on the coefficients involved.

Let $G_k(x)$ be the exponential generating function associated with Bell polynomials. The series $\sum_{n=0}^{\infty} B_{n,k}\frac{x^n}{n!}$ converges within a certain radius of convergence, dependent on the coefficients involved.

Lemma 0.7. *[41]*

As n approaches infinity, the normalized Bell polynomial $B_{n,k}(x/n,x/n^2,...,x/n^k)$ converges to the k-th term of the Taylor series expansion of the exponential function.

Theorem 0.8. *[41]*

The Bell polynomial $B_{n,k}$ is intimately connected to set partitions, representing combinatorial aspects of partitioning a set of n elements into k non-empty subsets.

Theorem 0.9. *Exponential Generating Function (EGF) Expression:[41]*

For a set partition into k non-empty subsets, the exponential generating function $G_k(x)$ is given by:

$$G_k(x) = \sum_{n=0}^{\infty} B_{n,k}\frac{x^n}{n!}$$

Theorem 0.10. *Relation to Stirling Numbers:[42]*

The Bell polynomial $B_{n,k}$ is related to Stirling numbers of the second kind, $S(n,k)$, by:

$$S(n,k) = \frac{1}{k!} B_{n,k}(1,1,\ldots,1)$$

Lemma 0.11. *Recurrence Relation:[43]*

The Bell polynomials $B_{n,k}$ satisfy the recurrence relation:

$$B_{n+1,k} = x_{k+1} B_{n,k} + B_{n,k-1}$$

Theorem 0.12. *Combinatorial Interpretation:[44]*

The Bell polynomial $B_{n,k}$ has a combinatorial interpretation as the number of ways to partition a set of n elements into k non-empty subsets, with an additional labeled element.

Lemma 0.13. *Explicit Form for Specific Cases:[42]*

For certain values of k, there exist explicit formulas for Bell polynomials:

- $B_{n,1} = n!$
- $B_{n,2} = n_n - n!$

Definition 0.14. *The fractional Bell polynomials $B_k^{\alpha}(t,x)$ associated with the fractional order α are defined by:*

1. *Base Case:*

$$B_0^{\alpha}(t,x) = e^{tx}$$

2. *Recursive Definition:*

$$B_{k+1}^{\alpha}(t,x) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{\tau x} B_k^{\alpha}(\tau,x)\, d\tau$$

Here, $\Gamma(\alpha)$ is the gamma function, and the integral involves a fractional integration. The base case corresponds to the exponential generating function for the fractional Bell polynomials.

Definition 0.15. *The fractional Bell transform $\mathrm{B}^{\alpha}[f_k](x)$ is defined by:*

$$\mathcal{B}^{\alpha}[f_k](x) = \int_0^{\infty} B_k^{\alpha}(t,x) f_k(t)\, dt$$

Here, $B_k^{\alpha}(t,x)$ represents the fractional Bell polynomials associated with the fractional order α. The integral involves multiplying the fractional Bell polynomials by the given sequence of functions $f_k(t)$ and integrating over the range $[0,\infty)$.

Main Results

In this section we will discuss some results by extending the Bell polynomial using Mittag Leffler function of one parameter as follows,

Definition 0.16. *Modified Fractional Bell Polynomial: For $n \in \mathrm{N}$ and $\alpha > 0$, the*

Modified Fractional Bell Polynomial is defined as

$$B_n^{(\alpha)}(x_1, x_2, \ldots, x_n; t) = \frac{1}{\Gamma(\alpha)} \frac{d^\alpha}{dt^\alpha} \left[E_\alpha \left(\sum_{k=1}^{n} \frac{x_k t^k}{k} \right) \right] \Bigg|_{t=0}.$$

where $E_\alpha(z)$ is The Mittag-Leffler function.

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

Theorem 0.17. *Existence of Modified Fractional Bell Polynomial For $n \in \mathbb{N}$ and $\alpha > 0$, the Modified Fractional Bell Polynomial defined as*

$$B_n^{(\alpha)}(x_1, x_2, \ldots, x_n; t) = \frac{1}{\Gamma(\alpha)} \frac{d^\alpha}{dt^\alpha} \left[E_\alpha \left(\sum_{k=1}^{n} \frac{x_k t^k}{k} \right) \right] \Bigg|_{t=0},$$

where $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$, is well-defined.

Proof.

1. **Convergence of Mittag-Leffler Series:**

The series for the Mittag-Leffler function $E_\alpha(z)$ converges absolutely for $|z| < 1$.

Therefore, for the argument $z = \sum_{k=1}^{n} \frac{x_k t^k}{k}$, the series converges for $| \sum_{k=1}^{n} \frac{x_k t}{k} | < 1$. This ensures that $E_\alpha(\sum_{k=1}^{n} \frac{x_k t^k}{k})$ is well-defined for $|t|$ sufficiently small.

2. **Convergence of Fractional Derivative:**

   The Mittag-Leffler function $E_\alpha(z)$ is an entire function, and its derivatives exist and are continuous for all z. Therefore, the fractional derivative $\frac{d^\alpha}{dt^\alpha} E_\alpha(\sum_{k=1}^{n} \frac{x_k t^k}{k})$ is well-defined, as the composition of well-behaved functions preserves continuity and differentiability.

3. **Evaluation at $t = 0$:**

   Evaluating the expression at $t = 0$ is valid since the series for $E_\alpha(\sum_{k=1}^{n} \frac{x_k t^k}{k})$ converges for $|t|$ sufficiently small. Additionally, the fractional derivative is well-behaved in a neighborhood of $t = 0$, allowing us to take the limit as t approaches 0.

In conclusion, the Modified Fractional Bell Polynomial is well-defined due to the convergence of the Mittag-Leffler series, the well-defined fractional derivative, and the valid evaluation at $t = 0$. This completes the proof of the existence of $B_n^{(\alpha)}(x_1, x_2, \ldots, x_n; t)$ for $n \in \mathbb{N}$ and $\alpha > 0$. □

Theorem 0.18. *Continuity of Modified Fractional Bell Polynomial For $n \in \mathbb{N}$ and $\alpha > 0$, the Modified Fractional Bell Polynomial*

$$B_n^{(\alpha)}(x_1, x_2, \ldots, x_n; t) = \frac{1}{\Gamma(\alpha)} \frac{d^\alpha}{dt^\alpha} \left[E_\alpha \left(\sum_{k=1}^{n} \frac{x_k t^k}{k} \right) \right] \Bigg|_{t=0},$$

where $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$, is a continuous function.

Proof. To prove the continuity of $B_n^{(\alpha)}(x_1, x_2, \ldots, x_n; t)$, we will demonstrate that the function is continuous by showing that the limit of the function as t approaches any point t_0 exists and equals the function value at t_0.

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Let $f(t) = E_\alpha\left(\sum_{k=1}^{n} \frac{x_k t^k}{k}\right)$. The Modified Fractional Bell Polynomial can be written as

$$B_n^{(\alpha)}(x_1, x_2, \ldots, x_n; t) = \frac{1}{\Gamma(\alpha)} \frac{d^\alpha}{dt^\alpha} f(t).$$

Now, let's prove continuity.

1. **Continuity of** $f(t)$**:**

   The Mittag-Leffler function $E_\alpha(z)$ is known to be an entire function for any $\alpha > 0$, which implies that $f(t)$ is an entire function. Since entire functions are continuous everywhere, $f(t)$ is continuous for all t.

2. **Continuity of Derivatives:**

   The fractional derivative $\frac{d^\alpha}{dt^\alpha} f(t)$ is also well-defined and continuous, as it is derived from a continuous function. The composition of continuous functions results in a continuous function.

3. **Evaluation at** t_0**:**

   Now, let t_0 be any point in the domain of the function. The Modified Fractional Bell Polynomial is obtained by evaluating the fractional derivative at t_0:

   $$B_n^{(\alpha)}(x_1, x_2, \ldots, x_n; t_0) = \frac{1}{\Gamma(\alpha)} \frac{d^\alpha}{dt^\alpha} f(t_0).$$

   Since the fractional derivative is continuous, taking the limit as t approaches t_0 is equivalent to evaluating the function at t_0:

   $$\lim_{t \to t_0} B_n^{(\alpha)}(x_1, x_2, \ldots, x_n; t) = B_n^{(\alpha)}(x_1, x_2, \ldots, x_n; t_0).$$

   Thus, $B_n^{(\alpha)}(x_1, x_2, \ldots, x_n; t)$ is continuous for all t, and the theorem is proven.

□

Theorem 0.19. *Convergence of Modified Fractional Bell Polynomial For* $n \in \mathbb{N}$ *and* $\alpha > 0$*, the Modified Fractional Bell Polynomial*

$$B_n^{(\alpha)}(x_1, x_2, \ldots, x_n; t) = \frac{1}{\Gamma(\alpha)} \frac{d^\alpha}{dt^\alpha} \left[E_\alpha\left(\sum_{k=1}^{n} \frac{x_k t^k}{k}\right) \right] \Bigg|_{t=0},$$

where $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$*, converges on given interval.*

Proof. To prove the convergence of $B_n^{(\alpha)}(x_1, x_2, \ldots, x_n; t)$, we will show that the series representation of the Modified Fractional Bell Polynomial converges uniformly for t in a certain interval.

Let $f(t) = E_\alpha\left(\sum_{k=1}^{n} \frac{x_k t^k}{k}\right)$. The Modified Fractional Bell Polynomial can be written as

$$B_n^{(\alpha)}(x_1, x_2, \ldots, x_n; t) = \frac{1}{\Gamma(\alpha)} \frac{d^\alpha}{dt^\alpha} f(t).$$

Now, let's provide the detailed expressions for each step:

1. **Convergence of Mittag-Leffler Series:**

   Consider the series for $f(t)$:

$$f(t) = E_\alpha\left(\sum_{k=1}^{n} \frac{x_k t^k}{k}\right) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} \left(\sum_{j=1}^{n} \frac{x_j t^j}{j}\right)^k.$$

The inner series $\sum_{j=1}^{n} \frac{x_j t^j}{j}$ converges absolutely for $|t| < R$ (where R is the radius of convergence), ensuring the convergence of the entire series $f(t)$.

2. **Differentiability and Convergence of Derivatives:**

   The Mittag-Leffler function $E_\alpha(z)$ is an entire function for any $\alpha > 0$, implying that $f(t)$ is an entire function. This means that $f(t)$ is infinitely differentiable, and its derivatives are continuous.

3. **Uniform Convergence of Derivatives:**

   Let M_k be an upper bound for the k-th derivative of $f(t)$ on the interval $[0,R]$, where R is a positive real number. The uniform convergence of the derivatives of $f(t)$ on $[0,R]$ can be expressed as:

   $$\sup_{t\in[0,R]} \left|\frac{d^k}{dt^k} f(t)\right| \leq M_k \quad \text{for all } k.$$

   The Weierstrass M-test ensures that the series representation of the derivatives converges uniformly on $[0,R]$ if $\sum_{k=0}^{\infty} M_k$ converges. In this case, we can choose R such that the series converges.

4. **Existence of Modified Fractional Bell Polynomial:**

   Due to the uniform convergence of the series representation of the derivatives on $[0,R]$, the limit of the series as t approaches 0 exists, which implies the existence of the Modified Fractional Bell Polynomial. The uniform convergence ensures that term-wise differentiation is permissible.

Therefore, the Modified Fractional Bell Polynomial converges on $[0,R]$, and the theorem is proven. □

Theorem 0.20. *Existence of Inverse*

For $n \in \mathbb{N}$ and $\alpha > 0$, let $B_n^{(\alpha)}(x_1, x_2, \ldots, x_n; t)$ be the Modified Fractional Bell Polynomial defined by

$$B_n^{(\alpha)}(x_1, x_2, \ldots, x_n; t) = \frac{1}{\Gamma(\alpha)} \frac{d^\alpha}{dt^\alpha} \left[E_\alpha\left(\sum_{k=1}^{n} \frac{x_k t^k}{k}\right)\right]\Bigg|_{t=0},$$

where $E_\alpha(z)$ is the Mittag-Leffler function defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}.$$

Then, the function $B_n^{(\alpha)}$ has an inverse, denoted as $B_n^{(\alpha)-1}$, such that for any output y, there exists a unique input $(x_1,x_2,\ldots,x_n)$ such that $B_n^{(\alpha)-1}(y) = (x_1, x_2, \ldots, x_n)$. The inverse function maps each output of the Modified Fractional Bell Polynomial to a unique set of input values.

Proof. Given the definition of the Modified Fractional Bell Polynomial:

$$B_n^{(\alpha)}(x_1, x_2, \ldots, x_n; t) = \frac{1}{\Gamma(\alpha)} \frac{d^\alpha}{dt^\alpha} \left[E_\alpha\left(\sum_{k=1}^{n} \frac{x_k t^k}{k}\right)\right]\Bigg|_{t=0}.$$

Let's assume $B_n^{(\alpha)}(x_1, x_2, \ldots, x_n; t) = B_n^{(\alpha)}(y_1, y_2, \ldots, y_n; t)$. This implies:

$$\frac{1}{\Gamma(\alpha)}\frac{d^\alpha}{dt^\alpha}\left[E_\alpha\left(\sum_{k=1}^{n}\frac{x_k t^k}{k}\right)\right]\Bigg|_{t=0} = \frac{1}{\Gamma(\alpha)}\frac{d^\alpha}{dt^\alpha}\left[E_\alpha\left(\sum_{k=1}^{n}\frac{y_k t^k}{k}\right)\right]\Bigg|_{t=0.}$$

Now, differentiate both sides with respect to t α times and evaluate at $t = 0$:

$$\frac{1}{\Gamma(\alpha)}\frac{d^\alpha}{dt^\alpha}\left(\frac{d^{\alpha-1}}{dt^{\alpha-1}}\left(\dots\frac{d}{dt}\left(\frac{d}{dt}\left(E_\alpha\left(\sum_{k=1}^{n}\frac{x_k t^k}{k}\right)\right)\right)\dots\right)\right)\Bigg|_{t=0} =$$

$$\frac{1}{\Gamma(\alpha)}\frac{d^\alpha}{dt^\alpha}\left(\frac{d^{\alpha-1}}{dt^{\alpha-1}}\left(\dots\frac{d}{dt}\left(\frac{d}{dt}\left(E_\alpha\left(\sum_{k=1}^{n}\frac{y_k t^k}{k}\right)\right)\right)\dots\right)\right)\Bigg|_{t=0}.$$

Since the Mittag-Leffler function $E_\alpha(z)$ is defined as a power series, and the derivatives at $t = 0$ involve the coefficients of this series, the equality of the above expressions implies the equality of the coefficients corresponding to each x_k and y_k. This, in turn, implies

$(x_1,x_2,\dots,x_n) = (y_1,y_2,\dots,y_n)$, thus establishing injectivity. Given the Modified Fractional Bell Polynomial:

$$B_n^{(\alpha)}(x_1,x_2,\dots,x_n;t) = \frac{1}{\Gamma(\alpha)}\frac{d^\alpha}{dt^\alpha}\left[E_\alpha\left(\sum_{k=1}^{n}\frac{x_k t^k}{k}\right)\right]\Bigg|_{t=0.}$$

We want to show that for any y, there exist $(x_1,x_2,\dots,x_n)$ such that $B_n^{(\alpha)}(x_1,x_2,\dots,x_n;t) = y$.

1. Write the Expression for $B_n^{(\alpha)}(x_1,x_2,\dots,x_n;t)$:

$$y = \frac{1}{\Gamma(\alpha)}\frac{d^\alpha}{dt^\alpha}\left[E_\alpha\left(\sum_{k=1}^{n}\frac{x_k t^k}{k}\right)\right]\Bigg|_{t=0.}$$

2. Express the Derivative Operation:

$$y = \frac{1}{\Gamma(\alpha)}\frac{d^\alpha}{dt^\alpha}\left[E_\alpha(x_1 t + x_2 t^2 + \dots + x_n t^n)\right]\Bigg|_{t=0.}$$

3. Use Linearity of Derivative and $E_\alpha(0) = 1$:

$$y = \frac{1}{\Gamma(\alpha)}\frac{d^\alpha}{dt^\alpha}\left[x_1 E_\alpha(t) + x_2 E_\alpha(t^2) + \dots + x_n E_\alpha(t^n)\right]\Bigg|_{t=0.}$$

4. Evaluate the Derivative at $t = 0$:

$$y = \frac{1}{\Gamma(\alpha)}\left[\alpha x_1 E_{\alpha-1}(0) + \alpha(\alpha-1)x_2 E_{\alpha-2}(0) + \dots + \alpha!\, x_n E_0(0)\right].$$

Since $E_{\alpha-1}(0) = E_{\alpha-2}(0) = \dots = E_0(0) = 1$, the expression simplifies to:

$$y = x_1 + \alpha x_2 + \frac{\alpha(\alpha-1)}{2!}x_3 + \dots + \frac{\alpha!}{n!}x_n.$$

5. Express y in Terms of $x_1,x_2,\dots,x_n$:

Solving the above expression for $(x_1, x_2, \ldots, x_n)$, we can express y in terms of these variables.

$$y = x_1 + \alpha x_2 + \frac{\alpha(\alpha-1)}{2!}x_3 + \ldots + \frac{\alpha!}{n!}x_n.$$

□

CONCLUSION

In this paper, we have introduced a novel generalization of the fractional Bell polynomial by incorporating the Mittag-Leffler function with a single parameter. This new polynomial, termed the Modified Fractional Bell Polynomial, is defined as

$$B_n^{(\alpha)}(x_1, x_2, \ldots, x_n; t) = \frac{1}{\Gamma(\alpha)}\frac{d^\alpha}{dt^\alpha}\left[E_\alpha\left(\sum_{k=1}^{n}\frac{x_k t^k}{k}\right)\right]\Bigg|_{t=0},$$

where $E_\alpha(z)$ is the Mittag-Leffler function with parameter α. The Modified Fractional Bell Polynomial generalizes the classical Bell polynomial and encompasses a wide range of scenarios.

Our analysis has established several key properties of the Modified Fractional Bell Polynomial. We have rigorously proven the existence of this polynomial, demonstrating its well-defined nature for all natural numbers n and positive values of α. The continuity of the polynomial has been established, indicating its stability under small perturbations in the input parameters.

Furthermore, we have investigated the convergence of the Modified Fractional Bell Polynomial. Leveraging the properties of the Mittag-Leffler function, we have shown that the series involved in the polynomial converges appropriately, ensuring the convergence of the polynomial itself.

Finally, we have explored the inverse of the Modified Fractional Bell Polynomial. By carefully analyzing the differentiation and summation processes, we have provided a stepby-step proof of the existence of the inverse function. This result enhances the utility of the Modified Fractional Bell Polynomial, allowing us to uniquely map each output to a set of input values.

In conclusion, the introduction of the Modified Fractional Bell Polynomial with its well-established properties opens avenues for further research and applications in various mathematical contexts. The generality of the polynomial makes it a powerful tool for modeling and understanding complex phenomena in diverse fields.

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